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An application of the refined Maslov–WKB technique to the one-dimensional Helmholtz equation

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Abstract. A refinement of the Maslov–WKB technique is used to determine the full asymptotic series of a solution to the one-dimensional Helmholtz equation near a turning point.

1. Introduction

We use a refinement of the Maslov–WKB method (the Maslov–WKB method as explained by Duistermaat (1974) and Guillemin and Sternberg (1977, ch 2)) to determine the full asymptotic series of a solution to the one-dimensional Helmholtz equation near a turning point. We assume that the one-dimensional Helmholtz equation

$$d^2\psi/dx^2 + \tau^2 f(x)\psi(x) = 0 \tag{1}$$

(τ a large parameter, $f(x)$ a function invertible near the turning point x_0) has an asymptotic solution near the turning point of the form

$$\psi(x) - \int A(x, p, \tau)\exp[i\tau(xp - S(p))]\,dp = O(\tau^{-\infty}), \tag{2}$$

where $A(x, p, \tau)$ and its derivatives are uniformly bounded, and $S(p)$ is such that $x - dS(p)/dp = 0$ determines the Lagrangian manifold of Maslov near the turning point (Eckmann and Seneor 1976). Then the eikonal, i.e.

$$|\nabla\phi|^2 - f(x) = 0, \tag{3}$$

associated with (1) is given by Maslov's Hamiltonian (Guillemin and Sternberg 1977, p 72)

$$H = p^2 - f(x). \tag{4}$$

2. Hamilton's equations

$$\dot{x} = dH/dp, \qquad \dot{p} = -dH/dx$$

define the Hamiltonian flow along Maslov's Lagrangian manifold with solution

$$t = \int_0^{x(t)} d\xi[4(f(\xi) - f(\theta)) + p^2(\theta)]^{-1/2}, \qquad p(t) = (f(x(t)))^{1/2}, \tag{5}$$

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where $\theta = x(0)$. The phase in the integral (2) is determined by noting

$$x = f^{-1}(p^2) = \mathrm{d}S/\mathrm{d}p;$$

$$\therefore \quad S = \int_{p(\theta)}^{p} f^{-1}(p^2)\, \mathrm{d}p,$$

where the initial condition is determined from (5). Hence

$$\phi = xp - S(p).$$

The amplitude $A(x, p, \tau)$ is the solution of the associated transport equation which is determined by carrying the differentiation (1) across the integral (2), i.e.

$$\int \exp[\mathrm{i}\tau(xp - S(p))][(\mathrm{i}\tau)^2(p^2 - f(x))A + 2\mathrm{i}\tau p\, \partial_x A + \partial_x^2 A]\, \mathrm{d}p = \mathrm{O}(\tau^{-\infty}). \tag{6}$$

The first term is Maslov's Hamiltonian (4). Expanding

$$p^2 - f(x) = p^2 - f\left(\frac{\mathrm{d}S}{\mathrm{d}p}\right) + \left(x - \frac{\mathrm{d}S}{\mathrm{d}p}\right)D(x, p) = \left(x - \frac{\mathrm{d}S}{\mathrm{d}p}\right)D(x, p),$$

where the remainder term is given by

$$D(x, p) = D = -\int_0^1 f\left(t\left(x - \frac{\mathrm{d}S}{\mathrm{d}p}\right) + \frac{\mathrm{d}S}{\mathrm{d}p}\right) \mathrm{d}t.$$

Substituting into (6),

$$\int \exp[\mathrm{i}\tau(xp - S(p))]\mathrm{i}\tau[-D\, \partial_p A - A\, \partial_p D + 2p\, \partial_x A + (1/\mathrm{i}\tau)\, \partial_x^2 A]\, \mathrm{d}p = \mathrm{O}(\tau^{-\infty}), \tag{7}$$

which is the requirement that the asymptotic series in τ of the integral be trivial. Hence we require

$$-D\, \partial_p A + 2p\, \partial_x A - A\, \partial_p D + (1/\mathrm{i}\tau)\, \partial_x^2 A = 0 \tag{8}$$

in a neighbourhood of the Lagrangian manifold (Guillemin and Sternberg 1977, p 18). Equation (8) leads to a transport equation in such a neighbourhood if we introduce the flow

$$\dot{x} = 2p, \qquad \dot{p} = -D(x, p) \tag{9}$$

(Gorman and Wells 1980). Equation (8) will hold in such a neighbourhood if we allow the asymptotic series

$$A(x, p, \tau) = \sum_k A_k(x, p)(\mathrm{i}\tau)^{-k}$$

to evolve according to the transport equation

$$\dot{A}_k - A_k\, \partial_p D + \partial_x^2 A_{k-1} = 0 \tag{10}$$

along the trajectories of (9). (Notice that, in general, the flow (9) is *not* the Hamiltonian flow.)

The full asymptotic series of the integral (2) is determined as the sum of the asymptotic series of integrals whose phase ϕ and amplitude A_k are determined above, i.e.

$$\int \exp[\mathrm{i}\tau(xp - S(p))]A(x, p, \tau)\,\mathrm{d}\tau \sim \sum_k \int \exp[\mathrm{i}\tau(xp - S(p))]A_k(x, p)(\mathrm{i}\tau)^{-k}\,\mathrm{d}p.$$

3. The coordinate transformations

At turning points (x_1, p_1), where

$$x_1 - \mathrm{d}S(p_1)/\mathrm{d}p = 0, \qquad \mathrm{d}^2S(p_1)/\mathrm{d}p_1^2 \neq 0,$$

the transformation

$$y = (p - p_1)|\tfrac{1}{2}\phi_p''(x_1, p_1) + \tfrac{1}{6}(p - p_1)\phi_p'''(x_1, p_1) + \ldots + (1/k!)(p - p_1)^{k-2}\phi_p^k(x_1, p_1) + \ldots|^{1/2}$$

transforms the integral

$$\int \exp(\mathrm{i}\tau\phi(x_1, p))A_k(x_1, p)\,\mathrm{d}p$$

to

$$\int \exp[\mathrm{i}\tau(\phi(x_1, p_1) \pm y^2)]g(x_1, y)\,\mathrm{d}y.$$

Here the non-constant exponential argument is now in normal (quadratic) form and $g(x_1, y) = \Sigma_m C_m(p_1)y^m$, where

$$C_m(p_1) = -\frac{1}{2\pi^2}\iint \frac{A_k(x_1, p + p_1)\,\mathrm{d}\xi\,\mathrm{d}p}{\xi^m p^{m-1}(\xi^2p^2 - \phi(x_1, p + p_1) + \phi(x_1, p_1))},$$

and the sign of $\pm y^2$ is the sign of $\phi_p''(x_1, p_1)$.

$$\therefore \quad \int \exp(\mathrm{i}\tau\phi(x, y))A_k(x, p)\,\mathrm{d}p = \exp(\mathrm{i}\tau\phi(x_1, p_1))\int \exp(\mathrm{i}\tau y^2)\sum_m C_m(p_1)y^m\,\mathrm{d}y,$$

where, for clarity, it is assumed $\phi_p''(x_1, p_1) > 0$. The asymptotic series of this last integral is determined using the stationary phase technique (Duistermaat 1973, p 23):

$$\int \exp\left(\frac{\mathrm{i}\tau\theta(a)y^2}{2}\right)g(y, a, \tau)\,\mathrm{d}y \sim \left(\frac{2\pi}{\tau}\right)^{1/2}|\theta(a)|^{-1/2}\exp\left(\mathrm{i}\frac{\pi}{4}\operatorname{sgn}\theta(a)\right)\sum_{l=0}^{\infty}\frac{1}{l!}D^l g(y, a, \tau)|_{y=0}\tau^{-l}, \qquad (11)$$

where $\theta(a)$ is non-singular, depending continuously on a, and

$$D = \frac{\mathrm{i}}{2\theta(a)}\frac{\mathrm{d}^2}{\mathrm{d}y^2}.$$

Applied to the above, for any A_k

$$\int \exp(\mathrm{i}\tau\phi(x_1, p))A_k(x_1, p)\, \mathrm{d}p$$

$$\sim\left(\frac{\pi}{\tau}\right)^{1/2} \exp\left[\mathrm{i}\tau\left(\frac{\pi}{4}+\phi(x_1, p_1)\right)\right] \sum_{l=0}^{\infty} \frac{1}{l} \frac{\mathrm{i}^l(2l)!\, C_{2l}(p_1)\tau^{-l}}{4^l}.$$

$$\therefore \quad \int \exp(\mathrm{i}\tau\phi(x, p))A(x, p, \tau)\, \mathrm{d}p \sim \sum_{k=0}^{\infty} (\mathrm{i}\tau)^{-k} \int \exp(\mathrm{i}\tau\phi(x, p))A_k(x, p)\, \mathrm{d}p.$$

At turning points (x_0, p_0), where

$$x_0 - \mathrm{d}S(p_0)/\mathrm{d}p = 0, \qquad \mathrm{d}^2S(p_0)/\mathrm{d}p^2 = 0, \qquad \mathrm{d}^3S(p_0)/\mathrm{d}p^3 \neq 0,$$

the transformation

$$y = (p - p_0)|\tfrac{1}{6}\phi_p'''(x_0, p_0) + \tfrac{1}{24}(p - p_0)\phi_p''''(x_0, p_0) + \ldots$$
$$+ (1/n!)(p - p_0)^{n-3}\phi_p^n(x_0, p_0) + \ldots|^{1/3}$$

transforms the integral

$$\int \exp(\mathrm{i}\tau\phi(x, p))A_k(x, p)\, \mathrm{d}p$$

to

$$\exp(\mathrm{i}\tau\phi(x_0, p_0)) \int \exp(\pm\mathrm{i}\tau y^3)h(x_0, y)\, \mathrm{d}y.$$

Here the exponential argument within the integral is the Thom fold, $h(x_0, y) = \Sigma_n C_n(p_0)y^n$, where

$$C_n(p_0) = -\frac{1}{2\pi^2} \iint \frac{A_k(x_0, p + p_0)\, \mathrm{d}\xi\, \mathrm{d}p}{\xi^n p^{n-1}(\xi^3 p^3 - \phi(x_0, p + p_0) + \phi(x_0, p_0))},$$

and the sign of $\pm y^3$ is the sign of $\phi_p'''(x_0, p_0)$.

$$\therefore \quad \int \exp(\mathrm{i}\tau\phi(x, p))A_k(x, p)\, \mathrm{d}p = \exp(\mathrm{i}\tau\phi(x_0, p_0)) \int \exp(\mathrm{i}\tau y^3)h(x_0, y)\, \mathrm{d}y,$$

where, for clarity, it is assumed $\phi_p'''(x_0, p_0) > 0$.

We define an operator $I(\tau)$ which carries functions $h(y)$ (which are bounded together with all their derivatives) to asymptotic series in $1/\mathrm{i}\tau$:

$$I(\tau)h(y) = \int_{-\infty}^{\infty} \exp(\mathrm{i}\tau y^3)h(y)\, \mathrm{d}y,$$

where we regard the integral as denoting its asymptotic series expansion as $\tau \to \infty$. Notice

$$\int_{-\infty}^{\infty} \exp(\mathrm{i}\tau y^3)h(y)\, \mathrm{d}y$$

$$= \alpha_0(h) \int_{-\infty}^{\infty} \exp(\mathrm{i}\tau y^3)\, \mathrm{d}y + \alpha_1(h) \int_{-\infty}^{\infty} \exp(\mathrm{i}\tau y^3)y\, \mathrm{d}y$$
$$+ \int_{-\infty}^{\infty} \exp(\mathrm{i}\tau y^3)\frac{y^2}{3}Rh(y)\, \mathrm{d}y,$$

where the last integral converges since the other three do, and where $\alpha_0(h)=h(0)$, $\alpha_1(h)=h'(0)$ and

$$Rh(y)=3\left(\frac{h(y)-h(0)-h'(0)y}{y^2}\right),$$

or alternatively

$$Rh(y)=3\int_0^1\int_0^1 h''(tsy)t\,\mathrm{d}t\,\mathrm{d}s.$$

Integrating by parts,

$$\int_{-\infty}^{\infty}\exp(\mathrm{i}\tau y^3)\frac{y^2}{3}Rh(y)\,\mathrm{d}y=-\frac{1}{\mathrm{i}\tau}\int_{-\infty}^{\infty}\exp(\mathrm{i}\tau y^3)\frac{\mathrm{d}(Rh(y))}{\mathrm{d}y}\,\mathrm{d}y.$$

(Notice that, if $h(y)$ is bounded, so are $\mathrm{d}(Rh(y))/\mathrm{d}y$ and all its derivatives (Guillemin and Sternberg 1977, pp 5–6).)

Define an operator T by

$$Th(y)=\mathrm{d}(Rh(y))/\mathrm{d}y.$$

$$\therefore\quad I(\tau)h(y)=\alpha_0(h)J_0(\tau)+\alpha_1(h)J_1(\tau)-(1/\mathrm{i}\tau)I(\tau)Th(y), \tag{12}$$

where

$$J_j(\tau)=\int_{-\infty}^{\infty}\exp(\mathrm{i}\tau y^3)y^j\,\mathrm{d}y,\qquad j=0,1,$$

are determined by contour integration:

$$J_0(\tau)=\tfrac{2}{3}\tau^{-1/3}\Gamma(\tfrac{1}{3})\cos\tfrac{1}{6}\pi,\qquad J_1(\tau)=\tfrac{2}{3}\mathrm{i}\tau^{-2/3}\Gamma(\tfrac{2}{3})\sin\tfrac{1}{3}\pi.$$

Now we may write (12) as

$$I(\tau)[1+(1/\mathrm{i}\tau)T]=J_0(\tau)\alpha_0+J_1(\tau)\alpha_1,$$

where 1 is the identity operator, and α_0 and α_1 are operators carrying functions to constants. When dealing with formal power series, the operator $1+(1/\mathrm{i}\tau)T$ has the (right) inverse $\Sigma_{l=0}^{\infty}(-\mathrm{i}\tau)^{-l}T^l$.

$$\therefore\quad I(\tau)=(J_0(\tau)\alpha_0+J_1(\tau)\alpha_1)\left(\sum_{l=0}^{\infty}(-\mathrm{i}\tau)^{-l}T^l\right);$$

hence

$$\int_{-\infty}^{\infty}\exp(\mathrm{i}\tau y^3)h(y)\,\mathrm{d}y$$
$$=J_0(\tau)\sum_{l=0}^{\infty}(-\mathrm{i}\tau)^{-l}T^lh(y)\Big|_{y=0}+J_1(\tau)\sum_{l=0}^{\infty}(-\mathrm{i}\tau)^{-l}\frac{\mathrm{d}}{\mathrm{d}y}(T^lh(y))\Big|_{y=0}.$$

$$\therefore\quad\int\exp(\mathrm{i}\tau\phi(x,y))A_k(x,p)\,\mathrm{d}p$$
$$\sim\exp(\mathrm{i}\tau\phi(x_0,p_0))\left(J_0(\tau)\sum_{l=0}^{\infty}(-\mathrm{i}\tau)^{-l}T^lh(y)\Big|_{y=0}\right.$$
$$\left.+J_1(\tau)\sum_{l=0}^{\infty}(-\mathrm{i}\tau)^{-l}\frac{\mathrm{d}}{\mathrm{d}y}(T^lh(y))\Big|_{y=0}\right)$$

and

$$\int \exp(i\tau\phi(x,p))A(x,p,\tau)\,dp \sim \sum_k (i\tau)^{-k} \int \exp(i\tau\phi(x,p))A_k(x,p)\,dp.$$

4. Example

If in (1) $f(x)$ is linear, then the flow (9) is identical with that of Maslov. If, however,

$$f(x) = E - x^2,$$

the Hamiltonian (4) becomes

$$H = p^2 + x^2 - E,$$

and (5) becomes explicitly (with $\phi = \nu \cos \gamma_0$, to deal with a point source of radiation)

$$x(\nu, t) = \alpha\nu \cos 2t + \beta\nu \sin 2t = \nu \cos(\gamma_0 - 2t),$$

$$p(\nu, t) = -\alpha\nu \sin 2t + \beta\nu \cos 2t = \nu \sin(\gamma_0 - 2t),$$

where $(\alpha, \beta) = (\cos \gamma_0, \sin \gamma_0)$ and $\nu = E^{1/2}$. From

$$x = (E - p^2)^{1/2} = dS/dp,$$

$$\phi(x, p) = xp - \tfrac{1}{2}[E \sin^{-1}(p/E^{1/2}) + p(E - p^2)^{1/2}].$$

The stationary phase condition $\partial\phi/\partial p = 0$, $\partial^2\phi/\partial p^2 \neq 0$ determines that, at all $x = (E - p^2)^{1/2}$, $p \neq 0$, the method of stationary phase applies. At $p = 0$, $\partial^2\phi/\partial p^2 = 0$, but $\partial^3\phi/\partial p^3 \neq 0$. From (9)

$$D = -\dot{p} = -x - (E - p^2)^{1/2},$$

and hence the transport equation (10) becomes

$$\dot{A}_k - A_k\, \partial_p[-x - (E - p^2)^{1/2}] + \partial_x^2 A_{k-1} = 0. \tag{13}$$

The most general solution is

$$A_0 = \text{fcn}\{[x + 2(E - p^2)^{1/2}]^{1/3}[x - (E - p^2)^{1/2}]^{2/3}\}[x + 2(E - p^2)^{1/2}].$$

For $p \neq 0$, the asymptotic series of

$$\int \exp(i\tau\phi(x, p))A_0(x, p)\,dp$$

is determined from (11); for $p = 0$, the asymptotic series is determined from (12). Solving the transport equation for additional A_k's determines additional integrals, the sum of whose asymptotic expansions determines the full asymptotic expansion of

$$\int A(x, p) \exp(i\tau\phi(x, p))\,dp.$$

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References

Duistermaat J J 1973 *Fourier Integral Operators* (New York: Courant Institute)
—— 1974 *Comm. Pure Appl. Math.* **27** 207–81
Eckmann J P and Seneor R 1976 *Arch. Rat. Mech. Anal.* **61** 153–73
Gorman A and Wells R 1980 *Q. Appl. Math.*
Guillemin V and Sternberg S 1977 *Geometric Asymptotics* (Providence, RI: American Mathematical Society)